Minkowski Operators in Shape Modelling

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Visualisation of geometric structures obtained from Minkowski sums and products lead to views of surface patches demonstrating certain unusual artistic and aesthetic values and can be regarded as objects that stimulate our brains to develop higher dimensional imagination.

1. Introduction

Operation of Minkowski sum of two point sets was introduced by Hermann Minkowski in 1903 during his close cooperation with David Hilbert at the university in Göttingen. Nowadays this set operation has been re-introduced in connection to finding geometric algorithms for description of specific geometric problems dealing with layout optimisation, planning trajectory of a robot rigid motion in the working space avoiding obstacles, in offsetting and dense packing, for determination of equidistant manifolds and for shape modeling and morphing purposes in computer graphics, e.g. [1] - [3].

The most common interpretation of Minkowski sum of two point sets is by means of vector sum of the position vectors of all points in the given sets.

Let us consider infinite sets of points A and B, which are smooth manifolds in $\mathbf{E}^n$ determined by vector maps

$$A: {}^1\mathbf{r}(u) = ({}^1r_1(u), {}^1r_2(u), \ldots, {}^1r_n(u)), u \in [\alpha, \alpha']$$

$$B: {}^2\mathbf{r}(v) = ({}^2r_1(v), {}^2r_2(v), \ldots, {}^2r_n(v)), v \in [\beta, \beta']$$

Minkowski sum C of curves A and B is defined by

$$C: {}^1\mathbf{r}(u) = ({}^1r_1(u) + {}^2r_1(v), {}^1r_2(u) + {}^2r_2(v), \ldots, {}^1r_n(u) + {}^2r_n(v)), (u,v) \in [\alpha, \alpha'] \times [\beta, \beta']$$

Minkowski sum of parabolic arc and ellipse both equally parametrised for $u = v$ is illustrated in Figure 1, on the left, while Minkowski sum of parabolic arc and lemniscate of Bernoulli on the right in Figure 1.

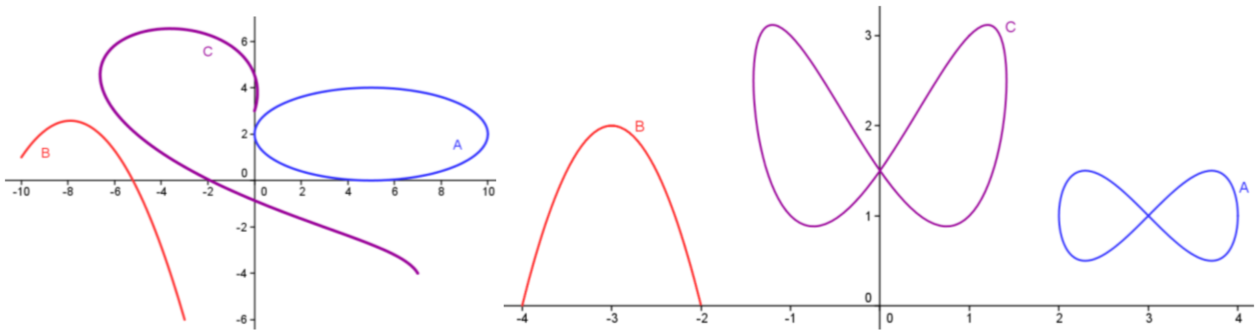

Figure 1. Minkowski sum of equally parameterized plane curve segments.

Some illustrations of surface patches that are generated as Minkowski sum of two curve segments in the space E^3 are presented in Figure 2. The basic curve segments are planar curves located in two mutually perpendicular planes, and from the left to the right they are the following: two ellipses (1), two parabolic arcs (2), versiére and shamrock curve (3), and finally asteroid and chain curve (4).

Surface patches that are Minkowski sums of the respective pairs of curves are presented in their orthographic axonometric views. Equations of curves are the following,

$$A: {}^1\mathbf{r}(u) = ({}^1a\cos u, {}^1b\sin u, 0), B: {}^2\mathbf{r}(v) = ({}^2a\cos v, 0, {}^2b\sin v), u, v \in [0, 2\pi] \tag{1}$$

$$A: {}^1\mathbf{r}(u) = ({}^1au, {}^1bu^2 + {}^1cu + {}^1d, 0), B: {}^2\mathbf{r}(v) = ({}^2av, 0, {}^2bv^2 + {}^2cv + {}^2d), u, v \in [0,1] \tag{2}$$

$$\begin{aligned} &A: {}^1\mathbf{r}(u) = ({}^1a(2u-1), 0, {}^1b/({}^1c + ({}^1a(2u-1))^2), u \in [0,1] \\ &B: {}^2\mathbf{r}(v) = (0, a\cos v\sin^2 v, a\cos^2 v\sin v), v \in [0, 2\pi] \end{aligned} \tag{3}$$

$$\begin{aligned} &A: {}^1\mathbf{r}(u) = (0, {}^1a(\cos 3u + 3(\cos u - 1)), -{}^1a(\sin 3u + 3\sin u)), u \in [0, 2\pi] \\ &B: {}^2\mathbf{r}(v) = ({}^2a(2v-1), 0, {}^2a/2(e^{{}^2b(2v-1)} + e^{-{}^2b(2v-1)}), v \in [0,1] \end{aligned} \tag{4}$$

where ${}^ia, {}^ib, {}^ic, {}^id$ are suitable constants determining form of curves.

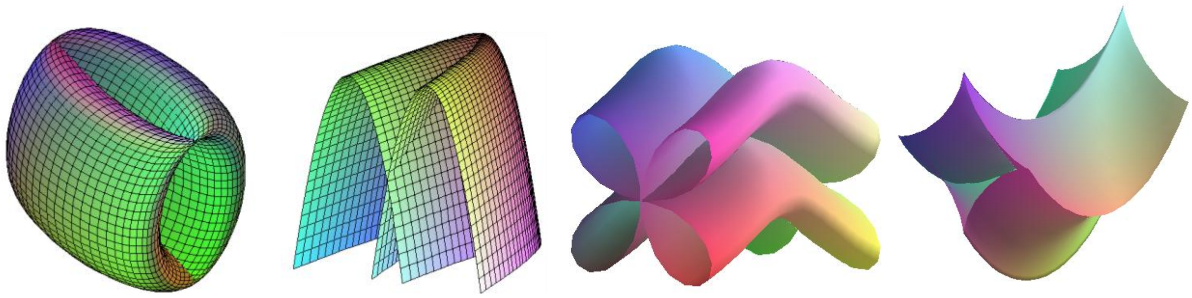

Figure 2. Minkowski sums of pairs of various curves in E^3.

In Figure 3, axonometric views of Minkowski sum of an ellipse and an elliptical conical helix from the space E^4 are illustrated with equation

$$\mathbf{r}(u,v) = ({}^1a\cos 2\pi u, {}^1b\sin 2\pi u + {}^2a(1-v)\cos 2\pi v, {}^2b(1-v)\sin 2\pi v, {}^2c2\pi v), (u,v) \in [0,1]^2.$$

Resulting surface patch is a manifold in the four-dimensional space, which can be projected orthogonally into all four possible three-dimensional subspaces determined by triples of coordinate axes. These orthographic views are surface patches in different 3-dimensional spaces with triples of coordinate axes depicted by their axonometric views.

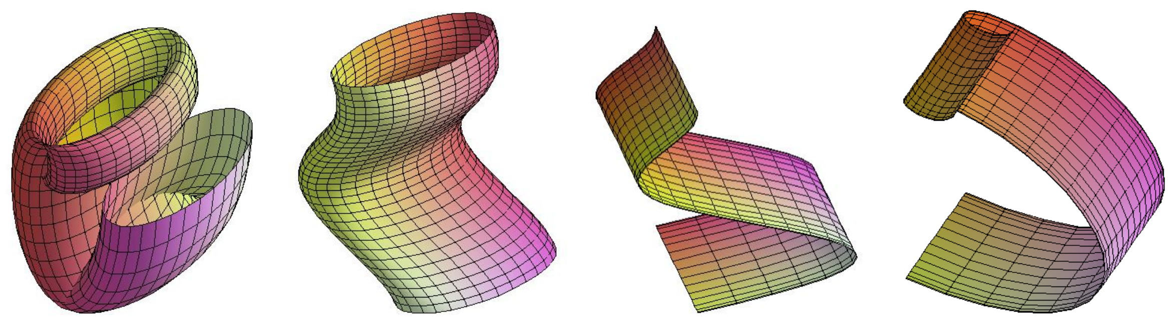

Figure 3. Views of Minkowski sum of ellipse and conical elliptical helix in E^4.

Minkowski linear combinations of two curve segments *A* and *B* are generated as Minkowski sums of *k*-multiple of curve *A* and *l*-multiple of curve *B* for any real numbers k and *l*, $C = k.A \oplus l.B$. Equation of linear combination of these two curves is

$$C: \mathbf{r}(u,v) = k.{}^1\mathbf{r}(u) + l.{}^2\mathbf{r}(v), (u,v) \in [\alpha, \alpha'] \times [\beta, \beta']$$

Minkowski linear combinations of an ellipse and parabolic arc positioned in one plane with a choice of two different specific values of coefficients k and l are illustrated in Figure 4.

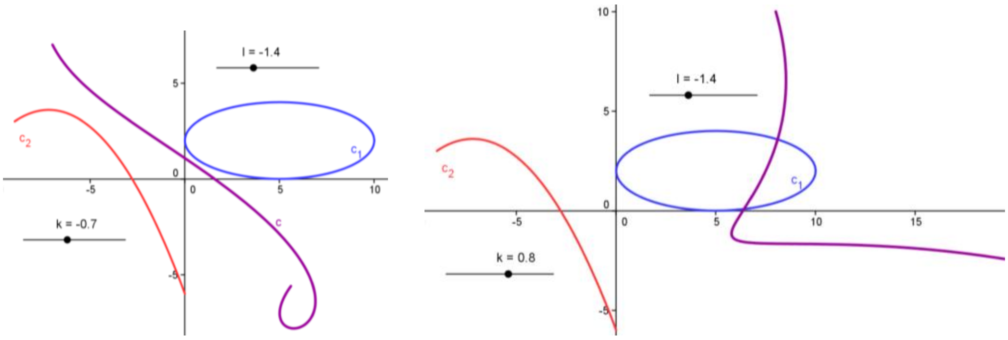

Figure 4. Minkowski linear combinations of ellipse and parabolic segment.

Surface patches illustrated in Figure 5 are Minkowski linear combinations of a lemniscate of Bernoulli

$$A:\ {}^1\mathbf{r}(u) = ({}^1a\cos u\sqrt{|\cos 2u|}, {}^1a\sin u\sqrt{|\cos 2u|}, 0), u \in [0, 2\pi], {}^1a \in R$$

and parabolic arc

$$B:\ {}^2\mathbf{r}(v) = ({}^2av, 0, bv^2 + cv + d), v \in [\beta, \beta'], {}^2a, b, c, d \in R$$

These two planar curves are positioned in two perpendicular planes in the 3-dimensional space. Their Minkowski linear combinations are surface patches *C* generated as Minkowski sums of *k*-multiple of curve *A* and *l*-multiple of curve *B* for any real numbers *k* and *l*, with equation (5)

$$\begin{aligned} C: \mathbf{r}(u,v) &= k\,{}^1\mathbf{r}(u) + l\,{}^2\mathbf{r}(v) = \\ &= ({}^1ak\cos u\sqrt{|\cos 2u|} + {}^2alv, {}^1ak\sin u\sqrt{|\cos 2u|}, l(bv^2 + cv + d)), (u,v) \in [0, 2\pi] \times [\beta, \beta'] \end{aligned} \tag{5}$$

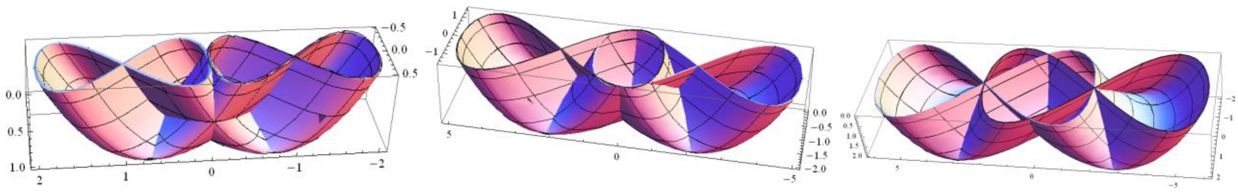

Figure 5. Minkowski linear combinations of two curve segments in 3D.

Minkowski linear combinations of two curve segments from higher dimensional spaces are well-defined and can model geometric figures in higher dimensions, details in [3]. Orthographic projection from 4D to different 3D subspaces can be easily obtained. Some examples of Minkowski linear combinations of curves, i.e. various surface patches in 4D are presented in Figure 6 in their orthographic views.

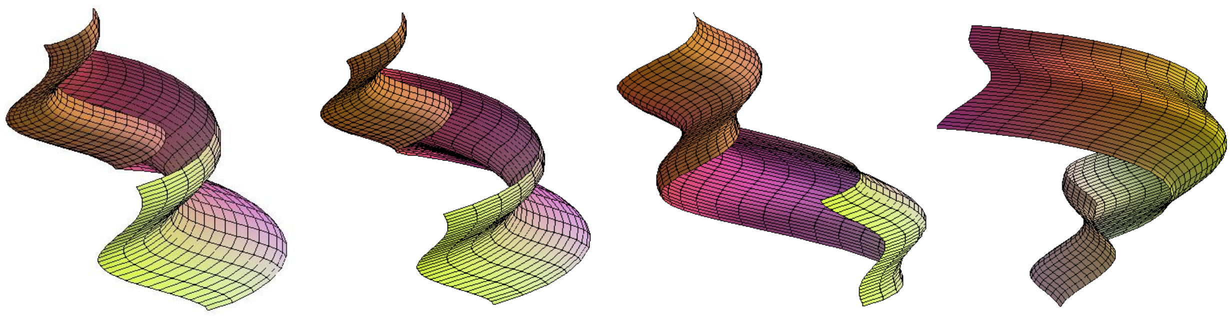

Figure 6. Orthographic views of Minkowski linear combinations of two helical segments in 4D.

Examples of Minkowski matrix combinations of two plane curve segments with the same parameter are curve segments presented in Figure 7. Minkowski matrix combinations of asteroid and versiére in perpendicular planes are mapped in figure 8, above (their Minkowski sum is in Figure 3 on the right), while Minkowski matrix combinations of parabolic arc and lemniscate of Bernoulli are below, for comparison with their Minkowski linear combinations in Figure 4.

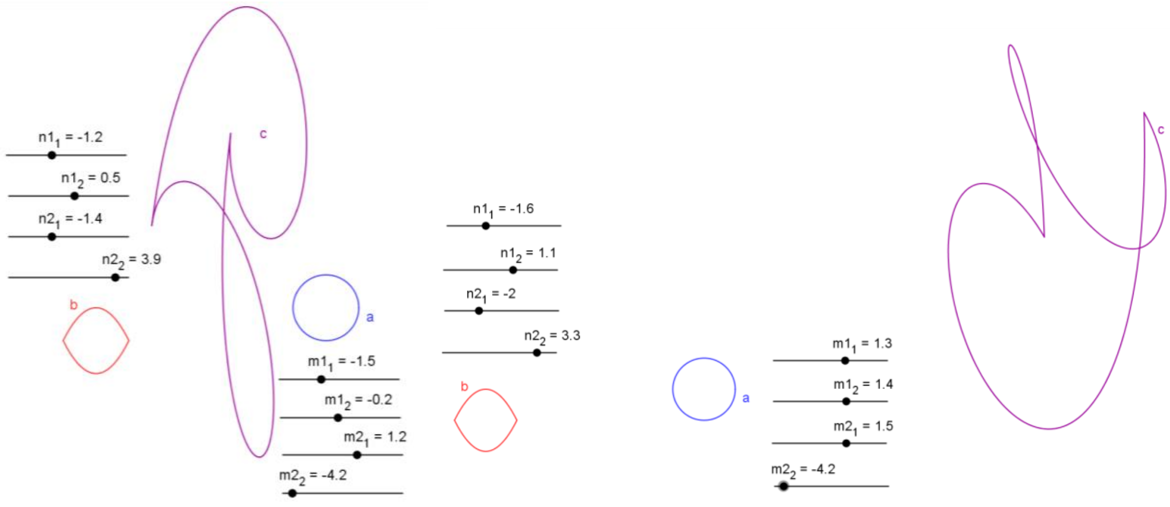

Figure 7. Minkowski matrix combinations of plane curve segments.

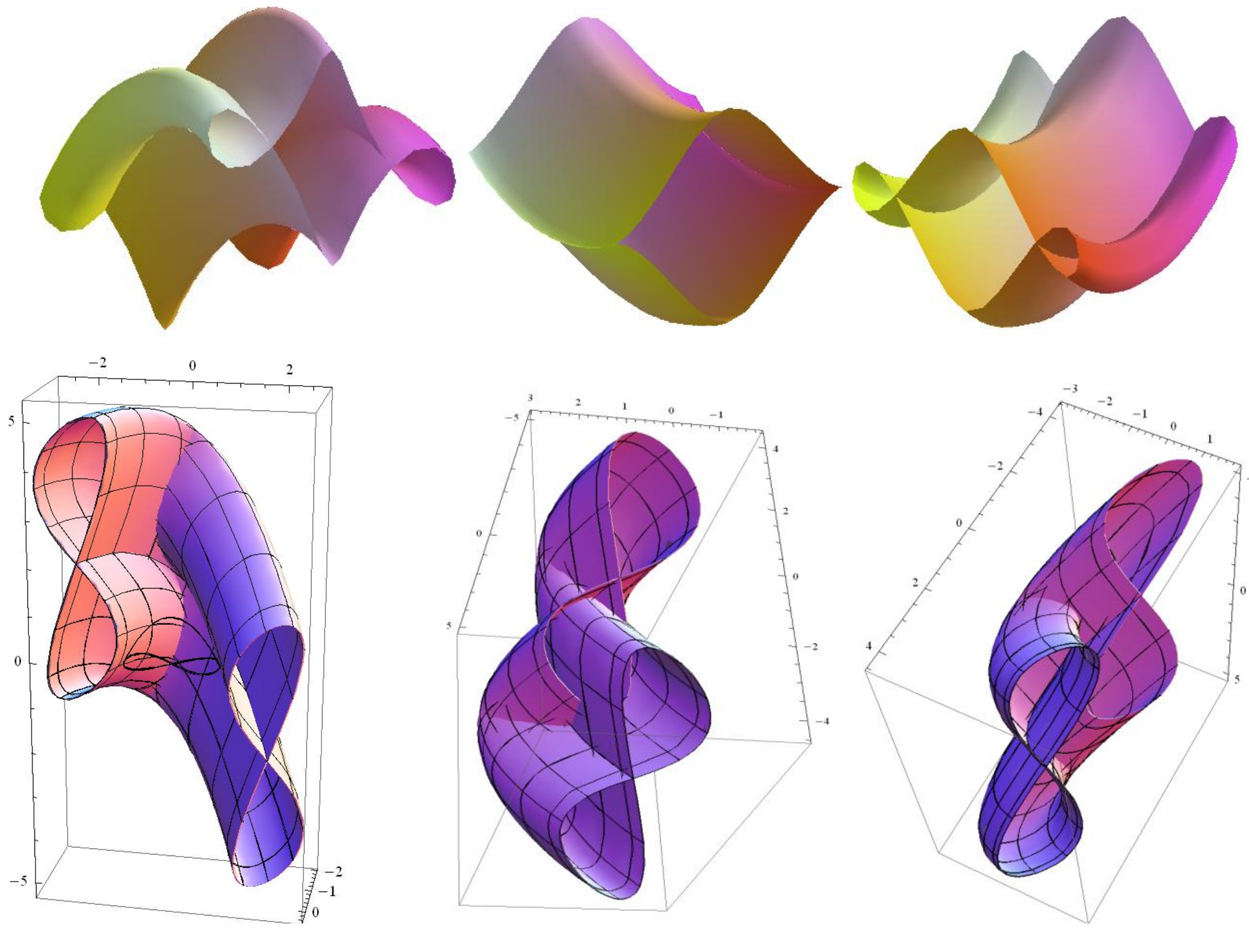

Figure 8. Minkowski matrix combinations of curves in E^3.

2. Minkowski product of point sets

Minkowski product of two curves is the surface patch determined by vector equation that is the wedge product of vector equations of the two curves. Considering curve segments A and B defined by their vector equations ${}^{1}\mathbf{r}(u), u \in [\alpha, \alpha']$, ${}^{2}\mathbf{r}(v), v \in [\beta, \beta']$, their Minkowski product is a surface patch C determined by equation

$$C : \mathbf{p}(u,v) = {}^{1}\mathbf{r}(u) \wedge {}^{2}\mathbf{r}(v), (u,v) \in [\alpha, \alpha'] \times [\beta, \beta']$$

Example in figure 9 below shows the Minkowski product of a parabolic arc and the lemniscate of Bernoulli, in comparison to their Minkowski sum above presented in Figure 5, left. Equation of this surface patch is

$$C : \mathbf{p}(u,v) = ({}^{1}a(bv^2 + cv + d)\sin u\sqrt{|\cos 2u|}, {}^{1}a(bv^2 + cv + d)\cos u\sqrt{|\cos 2u|}, {}^{1}a^2 av \sin u\sqrt{|\cos 2u|}),$$
$$(u,v) \in [0, 2\pi] \times [\beta, \beta']$$

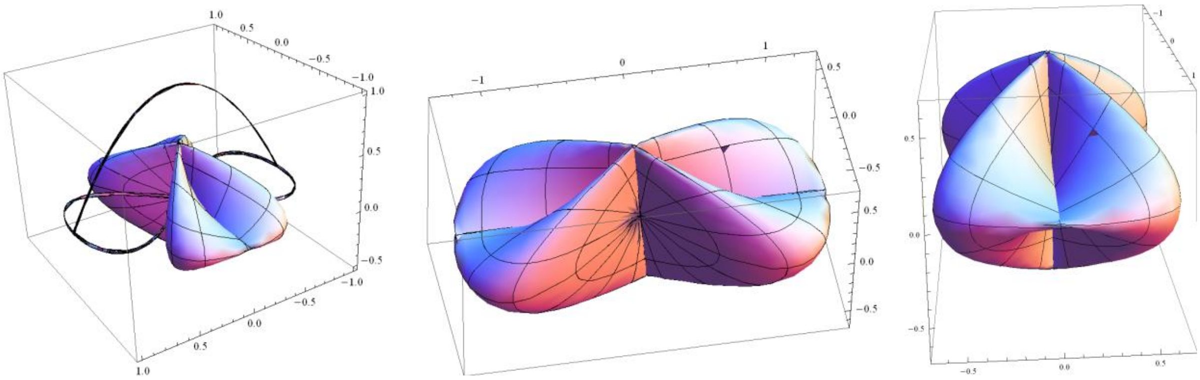

Figure 9. Minkowski product of two curve segments in 3D.

Views of the Minkowski product of differently parameterized shamrock curve and versiére that are located in different 3-dimensional subspaces of E^4 show a spectacular surface patch with equation (6) from the space $\boldsymbol{E}^4$ presented in different 3-D views in Figure 10.
Equation (6) of this surface has the following form

$$\mathbf{p}(u,v) = \begin{pmatrix} a(2u-1)\cos v \sin^2 v \\ \dfrac{a \cos v \sin^2 v}{(b + a(2u-1)^2)} \\ 0 \\ \dfrac{a \sin v \cos^2 v}{(b + a(2u-1)^2)} \end{pmatrix}^T, (u,v) \in [0,1] \times [0, 2\pi]. \qquad (6)$$

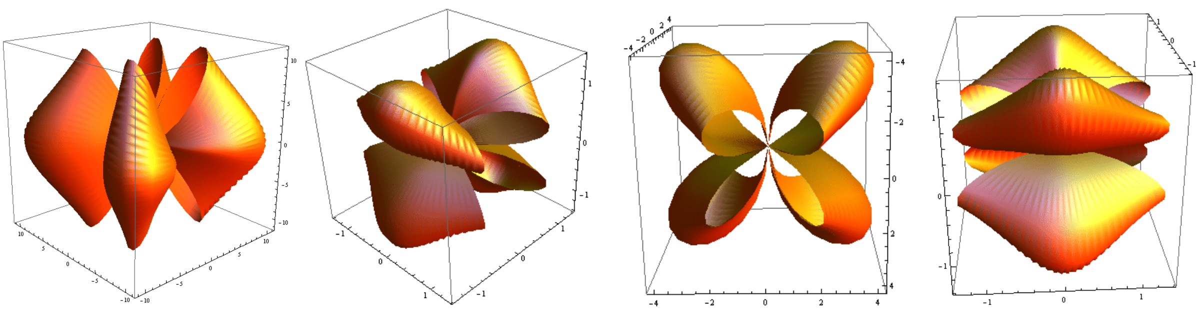

Figure 10. Views of Minkowski product of two curve segments in 4D.

3. Minkowski triples

Various Minkowski combinations of three point sets *A*, *B*, *C* can be introduced, in order to create unusual forms of point sets in $\mathbf{E}^n$ with specific properties determined by their generating principles or inherited from the original operands, smooth manifolds in 2 and even more dimensional spaces. Except for sums, combinations of sums and products depend on the order in which these operations are introduced.

Let ⊕ denote the Minkowski sum, ⊗ the Minkowski product, and consider:

A defined by ${}^1\mathbf{r}(u), u \in [\alpha, \alpha']$

B defined by ${}^2\mathbf{r}(v), v \in [\beta, \beta']$

C defined by ${}^3\mathbf{r}(w), w \in [\gamma, \gamma']$.

An example of surface generated as a Minkowski sum triple

$$A \oplus B \oplus C\text{: } \mathbf{sr}(u,v,w) = {}^1\mathbf{r}(u) + {}^2\mathbf{r}(v) + {}^3\mathbf{r}(w), (u,v,w) \in [\alpha,\alpha'] \times [\beta,\beta'] \times [\gamma,\gamma']$$

of three curve segments positioned in perpendicular planes in $\mathbf{E}^3$, parabolic arc, circle and lemniscate of Bernoulli and given by equation

$$\mathbf{sr}(u,v) = (\cos u + \cos v \sin v, \cos v + \sin v, \sin u + \sin^2 v), (u,v) \in [0,2\pi]^2 \qquad (7)$$

is presented in different views in figure 11.

A Minkowski linear mixed combinations (*A* ⊕ *B*) ⊗ *C* of three circles *A*, *B*, *C* located in perpendicular planes, while 2 of them are equally parameterized, are surface patches defined by

$$\mathbf{sm}(u,v,w) = ({}^1\mathbf{r}(u) + {}^2\mathbf{r}(v)) \wedge {}^3\mathbf{r}(w), (u,v,w) \in [\alpha,\alpha'] \times [\beta,\beta'] \times [\gamma,\gamma']$$

illustrated in Figure 14.

Minkowski product triple *A* ⊗ *B* ⊗ *C* is defined by

$$\mathbf{sp}(u,v,w) = ({}^1\mathbf{r}(u) \wedge {}^2\mathbf{r}(v)) \wedge {}^3\mathbf{r}(w), (u,v,w) \in [\alpha,\alpha'] \times [\beta,\beta'] \times [\gamma,\gamma']$$

Various forms of surfaces can be obtained especially for equal parameterisations, in which either $u = v$, $u = w$ or $v = w$.

For instance Minkowski product triples of three curve segments with equation

$$\mathbf{sp}(u,v) = (-\cos u \sin v + \sin^2 u, -\sin^2 u(\cos v + \cos u), \cos u \cos v + \sin v \cos^2 v), (u,v) \in [0,2\pi]^2 \qquad (8)$$

can be seen in figure 12, while Minkowski mixed triples of these curves with equation

$$\mathbf{sm}(u,v) = \begin{pmatrix} \sin u(3\cos v + \sin v) \\ \sin v(-\sin u \cos v + 3\cos u \sin v) \\ -\cos u(3\cos v + \sin v) \end{pmatrix}^T, (u,v) \in [0,2\pi]^2 \qquad (9)$$

are illustrated in Figure 13.

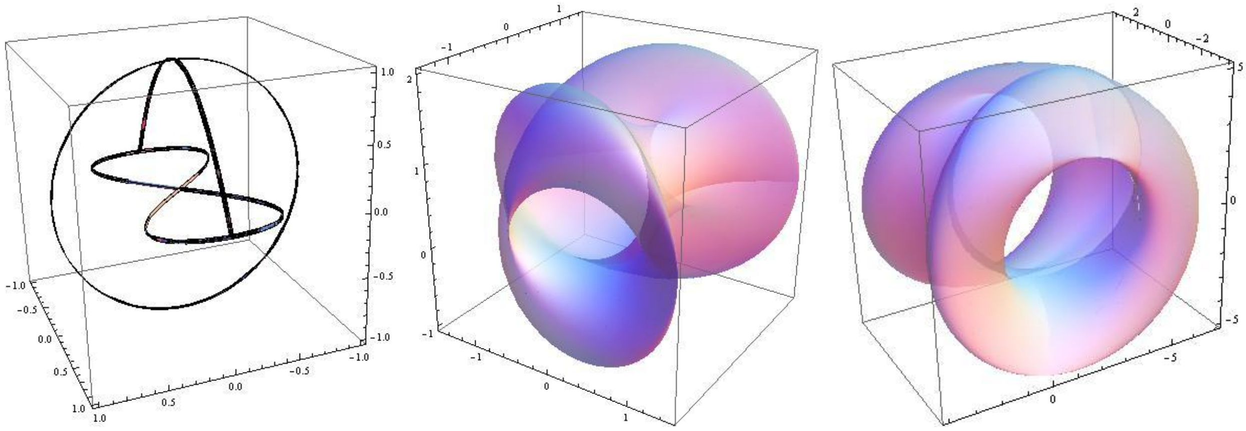

Figure 11. Views of Minkowski sum triple of three curve segments in 3D.

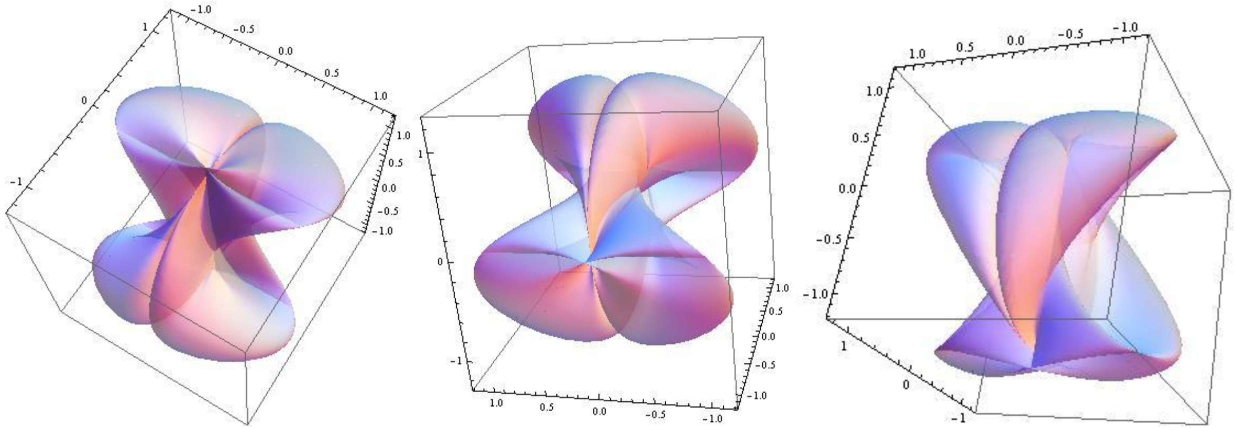

Figure 12. Views of Minkowski product triple of three curve segments in 3D.

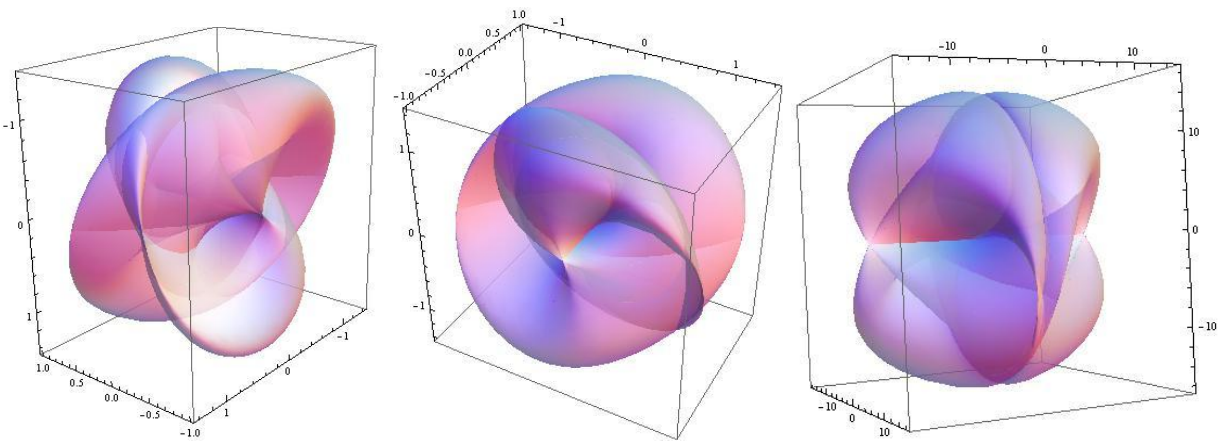

Figure 13. Views of Minkowski mixed triple of three curve segments in 3D.

Interesting shapes and forms of generated surfaces can be used for purposes of graphic design, in visualisations or in morphing. The underlying principles of Minkowski set operations provide a tool for the generation of both, synthetic visual and analytic representations. Thus the intrinsic geometric properties of the created objects can be studied using the methods of differential geometry and the specific properties inherited from the operand sets can be detected and defined. New objects are therefore interesting from both the aesthetic and the theoretical mathematical point of view.

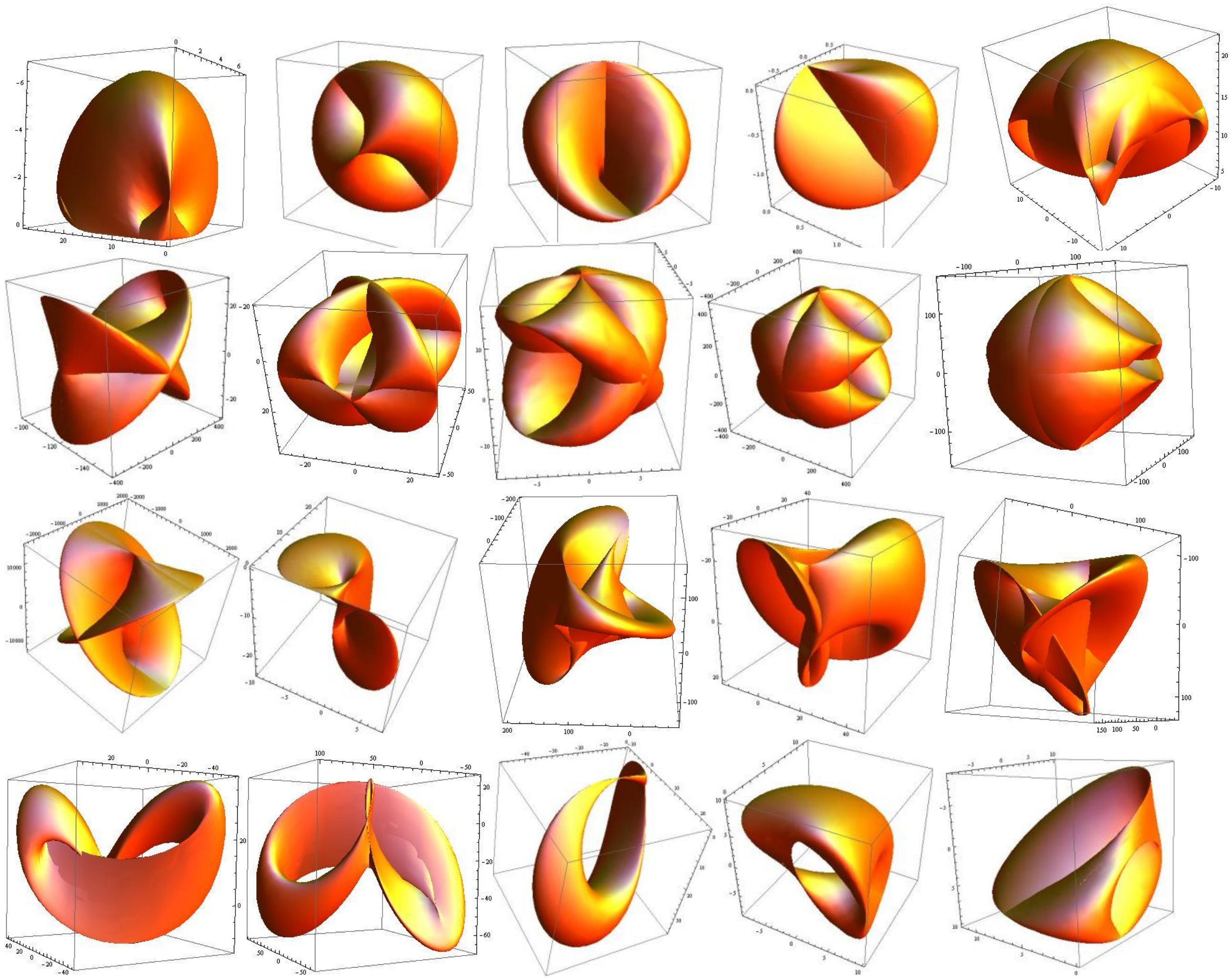

Figure 14. Minkowski linear triples of three circles.

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